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ON THE SINGULARITY OF ALTERNATING ELECTROMAGNETIC FIELDS IN THE VICINITY OF THE APEX OF A CONDUCTIVE WEDGE

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One problem of "technological hydrodynamics" [1] which has been actively developed recently is related to the noncrucible zone remelting of semiconducting materials in the alternating electromagnetic field of an inductor. A profound theoretical investigation of problems of heat- and mass-transfer in this process, which are characterized by the presence of unknown phase-transition boundaries (on which the physical properties of the medium are discontinuous), by the presence of free boundaries, and also by the nonlinearity of governing equations and boundary conditions, can be performed only by numerical methods. There are, however, some questions that can and must be previously studied by analytical methods. One of these questions concerning the behavior of alternating electromagnetic fields in the geometric-singularity region in the vicinity of the line of intersection of regions with different electrical conductivities of materials is discussed in the present paper.

1. The features of the problem, the unforseen difficulties arising during its investigation, and the methods of overcoming them are demonstrated here by means of the simplest two-dimensional formulation. A conductive wedge (conductivity σ) with apex angle $2\alpha_0$ and a rib which coincides with the z axis is considered. The wedge is placed in an alternating plane magnetic field (frequency ω) which is perpendicular to the z axis. In ambient space $\sigma = 0$, and the dielectric constant and the magnetic permeability are equal to unity. The characteristics of the electric and magnetic fields in the vicinity of the wedge apex both inside and outside it are to be determined.

In a quasi-steady approximation the vector potential $\mathbf{A}(r,\alpha,t) = A(r,\alpha)e^{i\omega t}\mathbf{e}_z$ describing the desired fields $\mathbf{E} = -(1/c)(\partial \mathbf{A}/\partial t)$ and $\mathbf{H} = \operatorname{rot}\mathbf{A}$ is determined from the problem

$$\Delta A^{(1,2)}(r,\alpha) - \frac{2i}{\delta^2}A^{(1,2)}(r,\alpha) = 0, \qquad \delta^{(1,2)} = \begin{cases} 1 & \text{for region 1,} \\ 0 & \text{for region 2;} \end{cases} \tag{1.1}$$

$$A^{(1)} = A^{(2)}\Big|_{\alpha=\alpha_0}, \quad \frac{\partial A^{(1)}}{\partial \alpha} = \frac{\partial A^{(2)}}{\partial \alpha}\Big|_{\alpha=\alpha_0}, \quad A^{(1)} = A^{(2)}\Big|_{\alpha=2\pi-\alpha_0}, \quad \frac{\partial A^{(1)}}{\partial \alpha} = \frac{\partial A^{(2)}}{\partial \alpha}\Big|_{\alpha=2\pi-\alpha_0}. \tag{1.2}$$

The study of the asymptotic behavior of the fields near the wedge apex does not include the characteristic scale of the wedge l and the boundary conditions for a distant closing surface. In Eqs. (1.1) and conditions (1.2), the variables $A^{(1,2)}$ and r are regarded as the dimensionless variables obtained using the scales $H_0 l$ and l, respectively. The superscripts here indicate that the function belongs either to region 1 (conducting) or to region 2 of free space (see the Fig. 1). The dimensionless skin-layer thickness $\delta = c(\sqrt{2\pi\sigma\omega l})^{-1}$ and the coefficients $\delta^{(1,2)}$ are introduced to obtain a common form for the equation in regions 1 and 2. Note that boundary conditions (1.2) are written for the case where the magnetic permeability of the wedge material is $\mu = 1$. This is done to study the pure influence only of the conductivity on the behavior of the fields, since $\mu \neq 1$ determines the character of the singularity near the apex even in a zeroth approximation and makes uninteresting the subsequent approximations which take into account the effect of conduction.

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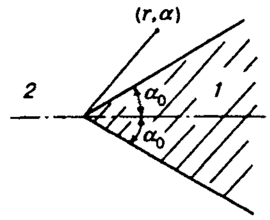

Fig. 1

Taking into account the solutions $r^{\nu+n}e^{\pm i(\nu+n)\alpha}$ and $J_{\nu+n}(((1-i)/\delta)r)e^{\pm i(\nu+n)\alpha}$ for the Laplace equation (region 2) and the heat conductivity equation (region 1), respectively, it seems quite natural that the solution of problem (1.1) and (1.2) for the region of $r \ll 1$ has the form of a series in powers of r:

$$A^{(1,2)}(r,\alpha) = \sum_{n=0}^{\infty} r^{\nu+n} B_n^{(1,2)}(\alpha), \tag{1.3}$$

where ν is a constant which is determined as an eigenvalue that ensures a nontrivial solution of the problem. Note that expansion (1.3) is the same as Meixner's expansion [2] if the latter is free from unjustified complications caused by simultaneous consideration of the components of the fields **E** and **H** instead of a single scalar function which is $A(r,\alpha)$ in the problem under consideration.

The study shows, however, that beginning with $n = 2$, expansion (1.3) cannot satisfy all boundary conditions (1.2), and this means that using this expansion it is not possible to obtain reliable data even for the initial approximations $n = 0$ and $n = 1$. This unexpected negative result causes one to seek a possible extension of the method of variable separation to obtain new solutions of the Laplace equation.

2. The search led to the following infinite set of solutions of the Laplace equation:

$$\varphi_n(r,\alpha) = r^{\nu+n} F_n(r,\alpha).$$

Here

$$F_n(r,\alpha) = \sum_{p=0}^{N(n)} C_{n,p}(\alpha)(\ln r)^p \qquad (n = 0,1,2,\ldots). \tag{2.1}$$

(The maximum number in the sum $p_{\max} = N(n)$ is determined for every specific problem and generally depends on n.)

Indeed, since

$$\Delta\varphi_n = r^{\nu+n-2}\Big\{ \sum_{p=0}^{N(n)} [C''_{n,p}(\alpha) + (\nu+n)^2 C_{n,p}(\alpha)](\ln r)^p$$

$$+ 2(\nu+n) \sum_{p=0}^{N(n)} pC_{n,p}(\ln r)^{p-1} + \sum_{p=0}^{N(n)} p(p-1)C_{n,p}(\alpha)(\ln r)^{p-2}\Big\},$$

where the expression in braces is a polynomial in $\ln r$, it is clear that if we make the coefficients of all powers $(\ln r)^p$ from $p = 0$ to $p_{\max} = N(n)$ vanish, we obtain $\Delta\varphi_n = 0$. Hence, the functions $C_{n,p}(\alpha)$ from (2.1) must satisfy the recurrent system of equations $[C''_{n,p} + (\nu+n)^2 C_{n,p}] + 2(\nu+n)(p+1)C_{n,p+1} + (p+1)(p+2)C_{n,p+2} = 0$ $[p = 0,1,\ldots,N(n)]$, which are solved sequentially starting with the maximum number $p_{\max} = N(n)$. This latter equation for $C_{n,p_{\max}}$ is homogeneous, and its solution is a combination of $\sin(\nu+n)\alpha$ and $\cos(\nu+n)\alpha$.

Obviously, the sum of solutions of the type of (2.1)

$$\varphi(r,\alpha) = r^{\nu} \sum_{n=0}^{\infty} r^n F_n(r,\alpha)$$

is also a solution of the Laplace equation.

In a similar manner, it is possible to construct a solution of the equation $\Delta A - (2i/\delta^2)A = 0$ as the series

$$A(r,\alpha) = \sum_{n=0}^{\infty} r^{\nu+n} F_n(r,\alpha), \qquad F_n(r,\alpha) = \sum_{p=0}^{N(n)} C_{n,p}(\alpha)(\ln r)^p. \tag{2.2}$$

Substituting (2.2) into the above equation, we obtain

$$\sum_{n=0}^{\infty} r^{\nu+n-2}\Big\{ \sum_{p=0}^{N(n)} [(C''_{n,p} + (\nu+n)^2 C_{n,p})(\ln r)^p + 2(\nu+n)pC_{n,p}(\ln r)^{p-1}$$
$$+ p(p-1)C_{n,p}(\ln r)^{p-2}] - \frac{2i}{\delta^2}\Lambda_n \sum_{p=0}^{N(n-2)} C_{n-2,p}(\ln r)^p \Big\} = 0.$$

Here

$$\Lambda_n = \begin{cases} 0 & \text{for } n = 0,1, \\ 1 & \text{for } n \geqslant 2. \end{cases}$$

From this it follows that (2.2) is actually a solution of the equation considered if the angular functions $C_{n,p}(\alpha)$ satisfy the equations

$$\begin{gathered}[C''_{n,p} + (\nu+n)^2 C_{n,p}] + 2(\nu+n)(p+1)C_{n,p+1} + (p+1)(p+2)C_{n,p+2} = \frac{2i}{\delta^2}\Lambda_n C_{n-2,p} \\ (p = 0,1,\ldots,N(n), \;\; n = 0,1,2,\ldots). \end{gathered} \tag{2.3}$$

It should be noted that the right-hand sides appear in (2.3) only beginning with $n = 2$, and the maximum value of the subscript p in $C_{n-2,p}$ is $N(n-2)$ and can be different from $N(n)$; if $N(n-2) < N(n)$, the right-hand sides disappear for $p > N(n-2)$.

It was assumed in solutions (2.1) and (2.2) that the number n is either equal to zero or take positive integer values. It is easy to verify that there is a second infinite set of solutions of the Laplace equation:

$$\varphi_n^{(2)} = r^{\nu-n}\Phi_n(r,\alpha), \quad \Phi_n(r,\alpha) = \sum_{p=0}^{N(n)} D_{n,p}(\alpha)(\ln r)^p \qquad (n = 0,1,\ldots),$$

where the angular functions satisfy the system of equations

$$D''_{n,p} + (\nu-n)^2 D_{n,p} + 2(\nu-n)(1+p)D_{n,p+1} + (1+p)(2+p)D_{n,p+2} = 0 \quad (p = 0,1,\ldots).$$

As solutions (2.2), the solutions $\varphi_n^{(2)}$ can be used as terms of a series for deriving a solution of the Helmholtz equation.

3. Let us use solutions of the type of (2.2) for initial problem (1.1) and (1.2). A successive examination of the approximations $n = 0,1,\ldots$ shows that as $N(n)$ one should take $N(n) = [n/2]$, i.e., the integer part of $n/2$, and, hence, the desired regular expansion can be written as

$$A^{1,2}(r,\alpha) = \sum_{n=0}^{\infty} r^{\nu+n} \sum_{p=0}^{[n/2]} C_{n,p}^{(1,2)}(\alpha)(\ln r)^p. \tag{3.1}$$

The functions $C_{n,p}^{(1,2)}(\alpha)$ satisfy Eqs. (2.3); one should only supplement the right-hand sides of these equations by the factor $\delta^{(1,2)}$ described above and write them as $(2i/\delta^2)\Lambda_n\delta^{(1,2)}C_{n-2,p}^{(1)}$.

We assume that the region of $0 < \alpha < \pi$ on one side of the symmetry plane of the wedge is the range of definition of the functions $A^{(1,2)}$ and consider two types of symmetry about this plane:

$$\text{(a)} \qquad A^{(1)}\,|_{\alpha=0} = 0, \quad A^{(2)}\,|_{\alpha=\pi} = 0; \qquad \text{(b)} \qquad \frac{\partial A^{(1)}}{\partial\alpha}\bigg|_{\alpha=0} = 0, \quad \frac{\partial A^{(2)}}{\alpha}\bigg|_{\alpha=\pi} = 0.$$

In both cases the coefficients of expansion (3.1) satisfy system (2.3) with corrected right-hand sides and the boundary conditions

$$C^{(1)}_{(n,p)}(0) = 0, \quad C^{(2)}_{n,p}(\pi) = 0, \quad C^{(1)}_{(n,p)}(\alpha_0) = C^{(2)}_{(n,p)}(\alpha_0), \quad \frac{dC^{(1)}_{(n,p)}}{d\alpha}(\alpha_0) = \frac{dC^{(2)}_{(n,p)}}{d\alpha}(\alpha_0); \tag{3.2a}$$

$$\frac{dC^{(1)}_{(n,p)}}{d\alpha}(0) = 0, \quad \frac{dC^{(2)}_{(n,p)}}{d\alpha}(\pi) = 0, \quad C^{(1)}_{(n,p)}(\alpha_0) = C^{(2)}_{(n,p)}(\alpha_0), \quad \frac{dC^{(1)}_{(n,p)}}{d\alpha}(\alpha_0) = \frac{dC^{(2)}_{(n,p)}}{d\alpha}(\alpha_0), \tag{3.2b}$$

which follows from (1.2) for (a) or (b) types of symmetry, respectively.

Let us consider the problem with symmetry (b). The zeroth term of expansion (3.1) is defined by the homogeneous equation $d^2C^{(1,2)}_{0,0}/d\alpha^2 + \nu^2 C^{(1,2)}_{0,0} = 0$ and the uniform boundary conditions (3.2b). A nontrivial solution for the problem exists for values of ν that satisfy the equation

$$\mathrm{Det}(\nu) = \sin \nu\pi = 0. \tag{3.3}$$

The minimum nonnegative eigenvalue $\nu_* = 0$, and, hence, the corresponding solution has the form $C^{(1,2)}_{0,0}(\alpha) = t_0 \cos \nu_*\alpha = t_0$ (t_0 is a constant which is the same for regions 1 and 2).

Since $\nu = \nu_* + 1$ satisfies Eq. (3.3), the term of series (3.1) corresponding to the number $n = 1$ also has the nontrivial solution $C^{(1,2)}_{1,0} = t_1 \cos(\nu_* + 1)\alpha$, which describes the uniform magnetic field perpendicular to the wedge symmetry plane with a dimensionless quantity specified by the indeterminate constant t_1.

Note that these results are also obtained from expansion (1.3). However, in this case, they can hardly be regarded as reliable since the next terms ($n \geqslant 2$) of expansion (1.3) cannot satisfy the boundary conditions.

The effect of conductivity starts to manifest itself beginning with the number $n = 2$. The corresponding term in (3.1)

$$A^{(1,2)}_2(r,\alpha) = r^2(C^{(1,2)}_{2,0}(\alpha) + C_{2,1}(\alpha)\ln r) \tag{3.4}$$

has a component with a logarithmic factor, and precisely this factor determines the character of the singularity in the vicinity of the apex of the conductive wedge. The angular functions from (3.4) satisfy Eqs. (2.3) which is written for $n = 2$, $p = 0$ and 1 as

$$\frac{d^2C^{(1,2)}_{2,0}}{d\alpha^2} + (\nu_* + 2)^2 C^{(1,2)}_{2,0} + 2(\nu_* + 2)C^{(1,2)}_{2,1} = \frac{2i}{\delta^2} C_{0,0}(\alpha)\delta^{(1,2)}; \tag{3.5}$$

$$\frac{d^2C_{2,1}}{d\alpha^2} + (\nu_* + 2)^2 C_{2,1} = 0. \tag{3.6}$$

Since $\mathrm{Det}(\nu_* + 2) = 0$, the homogeneous Eq. (3.6) admits the nontrivial solution $C^{(1,2)}_{2,1}(\alpha) = t_{2,1}\cos(\nu_* + 2)\alpha$, which satisfies boundary conditions (3.2b) for an arbitrary constant $t_{2,1}$ which enters into the solution of the inhomogeneous equation (3.5). Only owing to this are the functions $C^{(1,2)}_{2,0}$ able to satisfy the boundary conditions. The calculation result is as follows:

$$C^{(1)}_{2,0} = t^{(1)}_{2,0}\cos(\nu_* + 2)\alpha - t_{2,1}\alpha\sin(\nu_* + 2)\alpha + \frac{2i}{\delta^2}\frac{t_0}{4(\nu_* + 1)},$$

$$C^{(2)}_{2,0} = t^{(2)}_{2,0}\cos(\nu_* + 2)\alpha + t_{2,1}(\pi - \alpha)\sin(\nu_* + 2)\alpha.$$

Here the coefficients $t^{(1,2)}_{2,0}$ and $t_{2,1}$ are expressed in terms of t_0 as

$$t_{2,1} = \frac{2i}{\delta^2}\frac{t_0}{4\pi(\nu_* + 1)}\sin(\nu_* + 2)\alpha_0, \quad t^{(2)}_{2,0} - t^{(1)}_{2,0} = \frac{2i}{\delta^2}\frac{t_0}{4(\nu_* + 1)}\cos(\nu_* + 2)\alpha_0.$$

Hence, the function $C_{2,1}(\alpha)$ is uniquely determined by the constant t_0 of the zeroth approximation whereas $C^{(1,2)}_{2,0}(\alpha)$ are defined with accuracy up to the arbitrary components $\cos(\nu_* + 2)\alpha$, since boundary conditions (3.2b) impose restrictions not on the constants $t^{(1,2)}_{2,0}$ but only on their difference.

[Actually, there is a common result for all n which follows from Eqs. (2.3) and boundary conditions (3.2b): the functions $C_{n,0}^{(1,2)}(\alpha)$ are determined with accuracy up to the additive function $t_{n,0}\cos(\nu_* + n)\alpha$, where $t_{n,0}$ is an arbitrary (the same for regions 1 and 2) constant.] Thus, the obtained solution contains a set of free parameters which includes the coefficients t_0 and t_1 of the zeroth and first terms of the expansion, and also one of the two coefficients $t_{n,0}^{(1)}$ or $t_{n,0}^{(2)}$ of $\cos(\nu_* + n)(\alpha)$ in the expressions for $C_{n,0}^{(1,2)}(\alpha)$ for each subsequent term $n \geqslant 2$ of the expansion. These free constants are determined only by solution of the full problem.

Since the character of the singularity of the electromagnetic fields at the wedge apex due to its conductivity is mainly determined by the second term of the expansion, the expressions for the subsequent terms are not given here. It follows from expression (3.4) that the above-mentioned singularity consists in the appearance of the infinite derivatives $\partial H_\alpha/\partial r$ and $\partial H_r/\partial r$ at the point $r = 0$, while the fields H_α and H_r are continuous at this point. (Note that in [3] the eigenvalue $\nu = 0$ was not taken into account, which led to the invalid conclusion that singularities are absent.)

In the problem with the (a)-type symmetry, the minimum eigenvalue equals unity ($\nu_* = 1$). Therefore, the zeroth and first terms of expansion (3.1) subject to boundary conditions (3.2a) have the form

$$A_0^{(1,2)}(r,\alpha) = t_0 r \sin \nu_* \alpha, \quad A_1^{(1,2)}(r,\alpha) = t_1 r^2 \sin(\nu_* + 1)\alpha.$$

In a second approximation (taking into account the wedge conductivity), we have

$$A_2^{(1,2)} = r^3 (C_{2,0}^{(1,2)}(\alpha) + C_{2,1}(\alpha) \ln r),$$

where the logarithmic term appears with the factor r^3, making the solution free from the singularities described above. Obviously, in the general case, a superposition of solutions with the above-mentioned two types of symmetry holds.

Thus, the alternating electromagnetic fields in the vicinity of the line of intersection of regions with different conductivities are described by solutions of the type of (3.1). Therefore, in numerical studies of the full magnetohydrodynamic problem with phase transitions in which the conductivity has a discontinuity, a numerical scheme for the electrodynamic part of the problem should be developed with allowance for the weak singularities determined by the given solution.

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